AN ELEMENTARY PROBLEM OF RADIATION KINETICS WITH ARBITRARY INITIAL CONDITIONS

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§1. The nonstationary transfer of line radiation in a heated gas (a low-temperature plasma) is usually described using the kinetic equation for excited atoms [1]. Depending on the specific situation, problems then arise regarding the corresponding initial and boundary conditions. The features of the solution due to the different boundary conditions (different configurations of the volume of gas considered, the presence or absence of reflecting surfaces, etc.) have already been investigated to some extent, although the analysis, as a rule, refers to the stationary analog of the kinetic equation. As far as the initial conditions are concerned, it has been assumed up to now that at a certain initial instant of time ($t=t_0$) there is a uniform distribution of the excited atoms in the whole volume of gas. In many problems of radiation kinetics this assumption is not justified, for example, in a variety of cases of local energy dissipation, laser excitation, skin effect, etc. Thus, there is a need for a more general formulation of the problem of radiation kinetics (first of all, in its elementary form, which corresponds to the well-known Biberman–Holstein equation) for any initial distributions of the density of excited states.

§2. The change with time of the density of excited atoms n of the medium after the excitation has ceased is given by the equation [1, 2]

$$\tau_0 \frac{\partial n(\mathbf{r}, t)}{\partial t} = -n(\mathbf{r}, t) + \int_V n(\mathbf{r}', t) D(\mathbf{r}-\mathbf{r}')\, d\mathbf{r}', \tag{2.1}$$

where τ_0 is the spontaneous luminescence time; $D(\mathbf{r}-\mathbf{r}')$ is the probability that a quantum emitted at the point $\mathbf{r}'$ will be absorbed at the point $\mathbf{r}$,

$$D(\mathbf{r}-\mathbf{r}') = \frac{1}{4\pi H} \int_0^\infty \varkappa^2(\nu) \frac{\exp[-\varkappa(\nu)|\mathbf{r}-\mathbf{r}'|]}{|\mathbf{r}-\mathbf{r}'|^2}\, d\nu, \tag{2.2}$$

where $\varkappa(\nu)$ is the absorption factor; $H = \int_0^\infty \varkappa(\nu)\, d\nu$. In the case of cylindrical and spherical configurations of the luminous gas, Eq. (2.1) takes the form

$$\frac{\partial n(r, t)}{\partial t} = -n(r, t) + \int_0^1 n(r', t) G(r, r')\, dr', \tag{2.3}$$

where the time is measured in units of τ_0, while the coordinate is measured in units of the radius of the sphere or cylinder ($\tau_0 = R = 1$).

Equation (2.3) has a solution for any initial density distribution $n_0(r)$ of the excited atoms:

$$n(r, t) = \sum_{m=0}^{\infty} \exp[-\lambda_m t] C_m \varphi_m(r), \tag{2.4}$$

where λ_m and φ_m are the eigenvalues and eigenfunctions of the integral equation, connected by the following equation [3]:

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$$\varphi_m(r) - \int_0^1 \varphi_m(r') G(r, r') dr' = \lambda_m \varphi_m(r).$$

The coefficients C_m are determined by the initial condition

$$C_m = \int_0^1 n_0(r) \varphi_m(r) r^\alpha dr \Bigg| \int_0^1 \varphi_m^2(r) r^\alpha dr, \qquad \alpha = \begin{cases} 2 - \text{a sphere}, \\ 1 - \text{a cylinder}. \end{cases}$$

§3. For a numerical solution of Eq. (2.3) it is convenient to represent the kernel $G(r,r')$ in the form of an expansion in terms of the optical density parameter.

A Sphere. Integrating in Eq. (2.1) with respect to the nonradial coordinates, we obtain

$$G(r, r') = \frac{1}{2H} \frac{r'}{r} \int_0^\infty \varkappa^2(\nu) \{E_i[-\varkappa(\nu)(r + r')] - E_i[-\varkappa(\nu)|r - r'|]\} d\nu,$$

where $E_i(-x)$ is the integral exponential function. Using the expansion [4]

$$E_i(-x) = c + \ln x + \sum_{m=1}^{\infty} \frac{(-x)^m}{m m!},$$

we can obtain

$$G(r, r') = \frac{1}{2} \sum_{m=1}^{\infty} \frac{a_m k_0^{m+1} (-1)^m}{m m!} [(r + r')^m - |r - r'|^m] \frac{r'}{r} + a_0 k_0 \ln \left| \frac{r + r'}{r - r'} \right| \frac{r'}{r},$$

where k_0 is the absorption factor at the center of the line

$$a_m = \int_0^\infty \frac{\varkappa(\nu)}{H} \left(\frac{\varkappa(\nu)}{k_0}\right)^{m+1} d\nu. \tag{3.1}$$

A Cylinder. Introducing cylindrical coordinates, we obtain from Eq. (2.2)

$$G(r, r') = \frac{1}{2\pi H} \int_0^{2\pi} d\varphi \int_0^\infty \frac{\varkappa^2(\nu)}{q} d\nu \int_1^\infty \frac{\exp(-\varkappa(\nu) q x)}{x(x^2 - 1)^{1/2}} r' dx, \qquad q = (r^2 + r'^2 - 2rr' \cos\varphi)^{1/2}. \tag{3.2}$$

Using the Hubler integral representation for cylindrical functions

$$K_0(\xi) = \int_1^\infty \frac{\exp(-\xi x)}{(x^2 - 1)^{1/2}} dx,$$

Eq. (3.2) can be converted to the following form which is more convenient for expansion:

$$G(r, r') = \frac{1}{2\pi H} \int_0^{2\pi} d\varphi \int_0^\infty \varkappa^2(\nu) \left[\frac{\pi}{2} - \int_0^{\varkappa(\nu) q} K_0(\xi) d\xi\right] \frac{r'}{q} d\nu, \tag{3.3}$$

where $K_0(\xi)$ is the cylindrical function of imaginary argument (the Macdonald function) defined by the series expansion [4]

$$K_0(\xi) = \sum_{m=0}^{\infty} \frac{(\xi/2)^{2m}}{(m!)^2} \left\{\psi(m + 1) - \ln \frac{\xi}{2}\right\}, \qquad \psi(x) = d \ln \Gamma(x)/dx. \tag{3.4}$$

For an integral argument it is equal to $\psi(m + 1) = -c + \sum_{i=1}^{m} \frac{1}{i}$, where $c = 0.5772$ is Euler's constant.

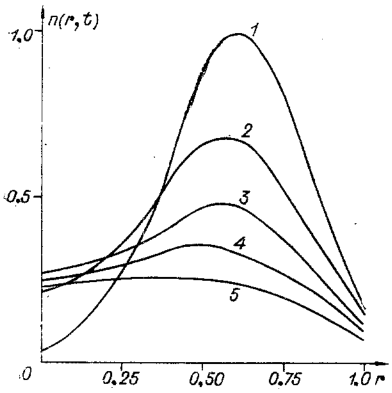

Fig. 1

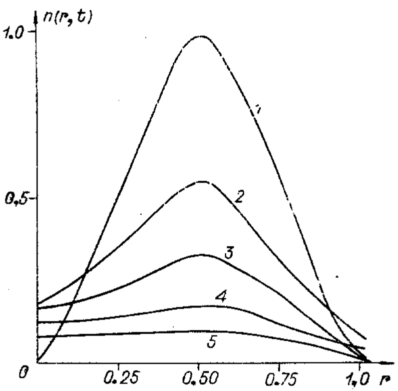

Fig. 2

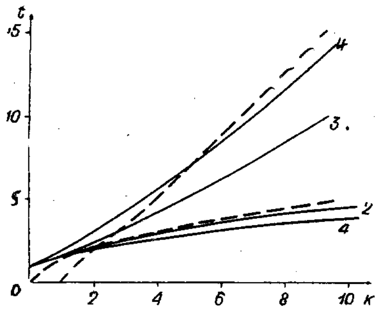

Fig. 3

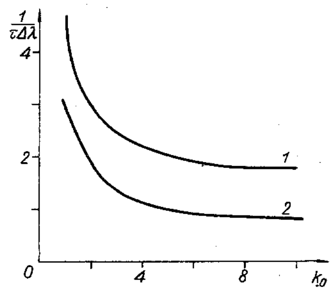

Fig. 4

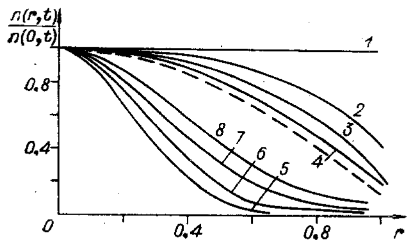

Fig. 5

Using Eq. (3.4), integration with respect to φ in Eq. (3.3) leads to the following expressions:

$$L = \frac{1}{4}\int_0^{2\pi}\frac{d\varphi}{q} = \begin{cases} \frac{1}{r} K(r'/r) & r > r' \\ \frac{1}{r'} K(r/r') & r < r', \end{cases}$$

where K(x) is the complete elliptic integral of the first kind;

$$S_\mu = \frac{1}{\pi}\int_0^{2\pi} q^{2\mu} d\varphi = 2\,|r^2 - r'^2|^\mu P_\mu\left(\frac{r^2 + r'^2}{|r^2 - r'^2|}\right);$$

$$U_\mu = \frac{1}{\pi}\int_0^{2\pi} q^{2\mu}\ln q\, d\varphi = \frac{1}{2}\frac{\partial S_\mu}{\partial \mu};$$

and $P_\mu(x)$ is the Legendre function.

For integral $\mu = m$, U_m and S_m can be calculated using the recurrent relation for P_μ [4]. After simple calculations we obtain

$$(2\mu + 1)P_\mu(x) = (\mu + 1)P_{\mu+1}(x) + \mu P_{\mu-1}(x),$$

$$P_0(x) = 1,\ P_1(x) = x,$$

$$S_{m+1} = \left(1 + \frac{m}{m+1}\right)(r^2 + r'^2)\, S_m - \frac{m}{m+1}|r^2 + r'^2|\, S_{m-1}.$$

$$S_0 = 2,\ S_1 = 2(r^2 + r'^2),$$

$$U_{m+1} = \left(1 + \frac{m}{m+1}\right)(r^2 + r'^2)\, U_m - \frac{m}{m+1}|r^2 - r'^2|\, U_{m-1} + \frac{1}{2(m+1)^2}(r^2 + r'^2)\, S_m - \frac{1}{2(m+1)^2}|r^2 - r'^2|\, S_{m-1},$$

$$U_0 = \ln\frac{r^2 + r'^2 + |r^2 - r'^2|}{2},$$

$$U_1 = (r^2 + r'^2)(U_0 + 1) - |r^2 - r'^2|.$$

Introducing the integrals with respect to frequency

$$b_m = \int_0^\infty \frac{\varkappa(\nu)}{H}\left(\frac{\varkappa(\nu)}{k_0}\right)^{m+1} \ln\frac{\varkappa(\nu)}{2}\,d\nu,$$

we can write the final expression for G(r, r'):

$$G(r, r') = r'a_0L + \sum_{m=0}^{\infty} \frac{\left(\frac{k_0}{2}\right)^{2m+1} r'}{(2m+1)(m!)^2} \left[(U_m - S_m/(2m+1) - S_m\psi(m+1))\,a_{2m+1} + S_m b_{2m+1}\right],$$

where a_m are defined by Eq. (3.1). In the case of Doppler and dispersion profiles of the absorption lines, the integrals a_m and b_m can be calculated accurately: for the Doppler profile

$$\varkappa(\nu) = k_0 \exp\left\{-\left(\frac{\nu-\nu_0}{\Delta\nu_D}\right)^2\right\}, \quad a_m = 1/(m+2)^{1/2},$$
$$b_m = a_m[\ln(k_0/2) - 1/2(m+2)];$$

and for the dispersion profile

$$\varkappa(\nu) = k_0 \frac{(\Gamma/2)^2}{(\nu-\nu_0)^2 + (\Gamma/2)^2},$$
$$a_m = \frac{(2m+2)!}{[(m+1)!]^2\, 2^{2m+2}}, \qquad b_m = a_m\left[\ln\frac{k_0}{8} + \sum_{i=1}^{m}\frac{1}{i(2i-1)}\right].$$

We will only consider these profiles below, since a consideration of more complex cases (for example, a Voigt profile) involves no essential difficulties and merely involves more complex calculations.

§4. In the section [0.1]2N+1 we choose equally spaced $r_1, \ldots, r_{2N+1}$ and we consider the corresponding discrete version of Eq. (2.3):

$$\frac{dn(r_i, t)}{dt} = -n(r_i, t) + \int_0^1 G(r_i, r')\,n(r', t)\,dr'. \tag{4.1}$$

Suppose the division step h=1/2N. Then in each section of length 2h we can replace n(r,t) approximately by a section of a parabola

$$n(r, t) = n(r_{i-1}, t)\frac{(r-r_i)(r-r_{i+1})}{2h^2} + n(r_i, t)\frac{(r-r_{i-1})(r-r_{i+1})}{h^2} + n(r_{i+1}, t)\frac{(r-r_{i-1})(r-r_i)}{2h^2}. \tag{4.2}$$

Integrating over the radius, Eq. (4.1) becomes the system of linear equations

$$\frac{\partial n_i(t)}{\partial t} = -\sum_j A_{ij} n_j(t), \qquad n_i(t) = n(r_i, t),\ A_{ij} = -\delta_{ij} + G_{ij}, \tag{4.3}$$

where G_{ij} is the result of integration of G with weights from Eq. (4.2). The integration was carried out on a computer. System (4.3) has the solution

$$n_i(t) = \sum_{m=1}^{2N+1} C_m \exp(-\lambda_m t)\, y_{im}, \tag{4.4}$$

where λ_m and y_{im} are the eigenvalues and corresponding eigenvectors of the matrix A_{ij}. The coefficients C_m can be found from the initial condition

$$n_i(0) = \sum_{m=1}^{2N+1} C_m y_{im}. \tag{4.5}$$

The first 2N+1 terms in Eq. (2.4) correspond to expression (4.4).

§5. As an example we will choose the initial distribution $n_0(r) = \exp\{-[(r-r_0)/\sigma]^2\}$. The attenuation pattern for spherical geometry, a Doppler profile, and $r_0=0.6$, $\sigma^2=0.1$, and $k_0=3$ is shown in Fig. 1. The result of calculations for the case of cylindrical geometry and a dispersion profile is shown in Fig. 2 ($r_0=0.5$, $\sigma^2=0.1$, and $k_0=3$). In both cases curve 1 corresponds to t=0, curve 2 corresponds to t=1, curve 3 corresponds to t=2, curve 4 corresponds to t=3, and curve 5 corresponds to t=4.

We will put $\min\{\lambda_i\} = \lambda$; $\tau = 1/\lambda$, corresponding to the eigenfunction y(r). It is seen from Eq. (2.4) that for fairly large times

$$n_i(t) = C y_i \exp[-t/\tau]. \tag{5.1}$$

Asymptotic expressions for τ for large optical thicknesses k_0 are given in [1] as they apply to a cylindrical geometry of the volume of gas

$$\tau = k_0(\pi \ln k_0)^{1/2}/1.60 \tag{5.2}$$

for a Doppler profile and

$$\tau = (\pi k_0)^{1/2}/1.115 \tag{5.3}$$

for a dispersion profile. (In our choice of variables, τ and k_0 are dimensionless quantities.) It is interesting to compare Eqs. (5.2) and (5.3) with the accurate values obtained in the present paper. The results of such a comparison are shown in Fig. 3. Curves 1 and 2 correspond to Doppler and dispersion profiles in the case of cylindrical geometry, the dashed curves correspond to Holstein's estimate, and curves 3 and 4 correspond to the same profiles in the case of spherical geometry. It is seen that if only a rough estimate of the quantity of approximately 10% is sufficient, one can use Eqs. (5.2) and (5.3) beginning at $k_0=5$; here, for comparison, we have given the values of τ relating to the case of a spherical configuration. It is seen that a sphere is de-excited more rapidly, as is, of course, obvious from qualitative considerations. In [5] the nonstationary equation (2.3) was solved for the case of a cylindrical configuration and a dispersion profile of the absorption line, and the case of large values of k_0 was investigated. As might have been expected, the results of the present paper agree with those of [5] for fairly large values of k_0.

Having a complete set of eigen numbers λ_m it is easy to estimate the time of emergence T on the asymptote (5.1):

$$T = \ln (C_1/C\varepsilon)/\Delta\lambda, \tag{5.4}$$

where ε is the relative difference between (5.1) and (4.5), $\Delta\lambda$ is the difference between λ and the eigen number closest to it, and C_1 is the corresponding coefficient in Eq. (4.5). Figure 4 shows the ratio T/τ as a function of the optical thickness in the case of a sphere (curve 1 is for a dispersion profile and curve 2 is for a Doppler profile). It is seen that for large values of the optical thickness T/τ =const.

This result must be understood in the sense that for large optical thicknesses, as shown in [5], all the eigen numbers are described by relations of the types (5.2) and (5.3). Numerical estimates are in good agreement with Eq. (5.4), and the value of T depends only slightly on the initial distribution. Initial distributions to which there corresponds a particle density in a certain small region are an exception. In this case the time of emergence on the asymptote increases considerably. Figure 5 shows graphs of the solution normalized to unity at the center of the sphere; the dashed curves represent the eigenfunction $y(r)$. Curves 1-4 are for solutions with a constant initial value ($\sigma=\infty$), and the time step is 2.5. Curves 5-8 represent similar solutions in which $\sigma^2=0.1$ and $r_0=0$. It is seen that the second group of solutions approaches the asymptote much more slowly. Note that according to Eq. (5.4), if $C_1=C$, then $n(T) \sim \varepsilon n(0)$, i.e., at the instant when Eq. (5.1) begins to be satisfied the volume of gas is already practically de-excited.

In conclusion, we note that the problem of radiation kinetics considered above for arbitrary initial conditions and its elementary formulation can also be formulated in a much more general form (see, for example, [6]). The approach to its solution in this case is not essentially different.

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